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1990 J. Phys. A: Math. Gen. 23 3543

(http://iopscience.iop.org/0305-4470/23/15/025)

J. Phys. A: Math. Gen. 23 (1990) 3453-3553. Printed in the UK

On similarity reductions of the three-wave resonant system to the Painlevé equations

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Received 28 December 1988

Abstract. The availability of more general similarity solutions than obtained previously is shown. Representations of the earlier known general complex similarity solutions in terms of Painlevé V and especially VI functions are given. Necessary comments and corrections of some published results regarding the Painlevé VI equation theory are made. A new Bäcklund transformation for the Painlevé VI equation is obtained.

Consider the three-wave resonant system:

$$u_{l,t}+c_l u_{l,x}+d_l u_{l,y}=\mathrm{i}u_j^* u_k^* \qquad 1\leqslant l,j,k\leqslant 3 \quad l\neq j\neq k \quad l\neq k. \tag{1}$$

The constants c_l and d_l, which represent corresponding group velocities, are assumed to be real and the equations themselves are put into order according to the condition

$$0<c_1<c_2<c_3.$$

The asterisks denote complex conjugation, and subscripts before the comma represent the number of the wavepacket and subscripts after the comma represent partial differentiation.

Similarity reductions of the system (1) were studied in [1-3]. Where three such reductions (I-III) were found. For the convenience of the reader they are listed below. However, there is a difference from the cited works. In the present paper more general reductions are written. Each one differs from the corresponding reduction given in [1-3] owing to the presence of additional real constants β_k $(k=1,2,3)$ related by the following condition:

$$\beta_1+\beta_2+\beta_3=0. \tag{2}$$

Besides that, we correct a mistake in [3] and, in case III, use more convenient variables than were used in [1]. The reductions are as follows.

Reduction I.

$$y_1=x-\tfrac{1}{2}(c_2+c_3)t \qquad y_2=y-\tfrac{1}{2}(d_2-d_3)t \qquad y_3=\gamma_1 y_1+\gamma_2 y_2$$

$$\gamma_1=\frac{c_3-c_2}{2\Delta}(2d_1-d_2-d_3) \qquad \gamma_2=\frac{c_2-c_3}{2\Delta}(2c_1-c_2-c_3)$$

$$\tau=y_3\exp[2\alpha(y_3-y_1)] \qquad \Delta=d_1(c_3-c_2)+d_2(c_1-c_3)+d_3(c_2-c_1)$$

$$u_1=\frac{a_1}{y_3}\exp[\mathrm{i}\beta_1(y_3-y_1)]v_1(\tau) \qquad a_1=\frac{\varepsilon(c_2-c_3)}{2} \qquad \varepsilon^2=1$$

$$u_2=a_2\exp[(\alpha+\mathrm{i}\beta_2)(y_3-y_1)]v_2(\tau) \qquad a_2=\sqrt{\gamma_2\Delta\alpha}$$

$$u_3 = a_3 \exp[(\alpha + i\beta_3)(y_3 - y_1)]v_3(\tau) \qquad a_3 = \varepsilon a_2.$$

Here α is an arbitrary real constant and the functions $v_k(\tau)$ satisfy the system of nonlinear ordinary differential equations (ODE):

$$\frac{i\beta_1}{2\alpha\tau} v_1 + v_1' = -i v_2^* v_3^* \qquad v_2' = \frac{i\varepsilon_0}{\tau} v_1^* v_3^* \qquad v_3' = -\frac{i\varepsilon_0}{\tau} v_1^* v_2^* \qquad \varepsilon_0 = \mathrm{sgn}(\alpha). \tag{3}$$

In the case $\beta_1 = 0$† in variables z, $w_k(z)$: $w_k(z) = 2v_k(\tau)$, $\tau = z^2$ the system (3) ($\varepsilon_0 = 1$) take the form

$$w_1' = -izw_2^* w_3^* \qquad w_2' = \frac{i}{z} w_1^* w_3^* \qquad w_3' = -\frac{i}{z} w_1^* w_2^*. \tag{4}$$

Purely imaginary solutions of the system (4)

$$w_j = i\tilde{w}_j \qquad \mathrm{Im}\, \tilde{w}_j = 0 \qquad j = 1, 2, 3$$

as was shown in [3], can be represented in terms of the third Painlevé equation

$$\phi'' = \frac{(\phi')^2}{\phi} - \frac{\phi'}{z} + \frac{C_1^2}{4}(\phi^3 - \phi^{-1}) \qquad \phi = \phi(z)$$

$$\tilde{w}_3 = C_1[\tfrac{1}{2} - \tfrac{1}{4}(\phi^2 + \phi^{-2})]^{1/2} \qquad \tilde{w}_2 = \sqrt{C_1^2 - \tilde{w}_3^2}.$$

In terms of a new variable $\varphi(z)$: $\phi^2 = -\exp(-i\varphi(z))$, these formulae may be rewirtten as follows:

$$(z\varphi')' + zC_1^2 \sin\varphi = 0 \qquad \tilde{w}_3 = C_1 \cos\frac{\varphi}{2} \qquad \tilde{w}_2 = \pm C_1 \sin\frac{\varphi}{2}. \tag{6}$$

Real solutions of equation (6) take part in the description of the solutions (5) of the system (4). Information concerning these solutions can be found in [4-6].

It turns out that general complex solutions of the system (4) can also be represented in terms of the third Painlevé equation. To show this, we introduce the notation

$$w_k = \rho_k \exp(i\varphi_k) \qquad k = 1, 2, 3 \qquad \psi = \varphi_1 + \varphi_2 + \varphi_3.$$

Then from system (4) we obtain

$$\rho_2^2 + \rho_3^2 = C_1^2 \qquad \rho_1\rho_2\rho_3 \cos\psi = C_2 \tag{7}$$

where C_1^2 and C_2 are the constants of integration. In the new notation the system (4) with the help of (7) may be rewritten in the form

$$\varphi_1' = -\frac{zC_2}{\rho_1^2} \qquad \varphi_2' = \frac{C_2}{z\rho_2^2} \qquad \varphi_3' = -\frac{C_2}{z\rho_3^2} \tag{8}$$

$$\rho_1\rho_1' = -zC_2 \tan\psi \qquad \rho_2\rho_2' = \frac{C_2}{z}\tan\psi \qquad \rho_3\rho_3' = -\frac{C_2}{z} \tan\psi \tag{9}$$

$$p_3'' + \frac{p_3'}{z} = \frac{1}{2}\left(\frac{1}{p_3} - \frac{1}{C_1^2 - p_3}\right)(p_3')^2 + \frac{2C_2^2}{z^2}\left(\frac{1}{p_3} - \frac{1}{C_1^2 - p_3}\right) + 2p_3(C_1^2 - p_3) \qquad p_3 = \rho_3^2. \tag{10}$$

By means of the transformation

$$p_3(z) = \frac{C_1^2}{1 - w(s)} \qquad s = z^2 \tag{11}$$

† In this case $\beta_2 = -\beta_3$ does not vanish in general (see (2)).

equation (10) can be brought into the standard form of the Painlevé V equation

$$w''=\left(\frac{1}{2w}+\frac{1}{w-1}\right)(w')^2-\frac{w'}{s}-\frac{1}{2}\left(\frac{C_2}{C_1^2}\right)^2\left(\frac{w-1}{s}\right)^2\left(w-\frac{1}{w}\right)-\frac{C_1^2 w}{2s}. \tag{12}$$

This particular case of the fifth Painlevé equation is connected by an invertible transformation to the third Painlevé equation (see, for instance, [7, 8]). Information about the solutions of (12) is given in [8, 9]†. Given a solution $w<0$ of equation (12) one may construct a solution of the system (4) in the following way. Equation (11) yields ρ_3^2, then the first equation in (7) gives ρ_2^2, and the second equation in (9) yields $\tan\psi$. When $\tan\psi$ has been obtained one can find $\cos^2\psi$ and from the second equation in (7) one obtains ρ_1^2. Finally, one can obtain phases φ_k by integration of (8).

Reduction II.

$$\tilde{y}_1=x-\tfrac{1}{3}(c_1+c_2+c_3)t \qquad \tilde{y}_2=y-\tfrac{1}{3}(d_1+d_2+d_3)t$$

$$a=[(-2d_1+d_2+d_3)\tilde{y}_1+(2c_1-c_2-c_3)\tilde{y}_2]/\Delta$$

$$b=[(2d_2-d_1-d_3)\tilde{y}_1+(-2c_2+c_1+c_3)\tilde{y}_2]/\Delta \qquad \tilde{z}=\frac{1-e^{-a}}{1-e^{-b}}.$$

Here Δ is defined by the same formula as in reduction I.

$$u_1=-\frac{1}{2i}e^{-ia\beta_3}(1-e^{-a})^{i\beta_1}\frac{\hat{w}_1(\tilde{z})}{\sinh(a/2)}$$

$$u_2=-\frac{1}{2i}(1-e^{-a})^{i\beta_2}\frac{\hat{w}_2(\tilde{z})}{\sinh(b/2)}$$

$$u_3=-\frac{1}{2i}(e^{a}-1)^{i\beta_3}e^{-a/2}\frac{\hat{w}_3(\tilde{z})}{(\tilde{z}-1)\sinh(b/2)}.$$

The functions $\hat{w}_k(\tilde{z})$ satisfy the following nonlinear system of ODE:

$$\begin{gathered}(\tilde{z}-1)\hat{w}_1'=\hat{w}_2^*\hat{w}_3^* \qquad -\frac{i\beta_3}{\tilde{z}-1}\hat{w}_3+\tilde{z}\hat{w}_3'=-\hat{w}_1^*\hat{w}_2^* \\ \tilde{z}(\tilde{z}-1)\hat{w}_2'+i\beta_2(\tilde{z}-1)\hat{w}_2=-\hat{w}_1^*w_3^*.\end{gathered} \tag{13}$$

In the case $\beta_2=\beta_3=0$ the system (13) was obtained in [1]. Real solutions of the system $\operatorname{Im}\hat{w}_k=0$, as was also shown in [1], are expressible in terms of solutions of the sixth Painlevé equation (PVI). It turns out that not only real but general complex solutions of the system (13) ($\beta_2=\beta_3=0$) can be expressed in such a way. This fact is the main result of the paper. It will be proved after we deal with the similarity reduction III.

Reduction III. In this case it is assumed that

$$d_1=d_2=d_3=0.$$

The similarity reduction is

$$u_k=(c_kt-x)^{-1+i\beta_k}\psi_k(\zeta) \qquad \zeta=\frac{x}{t} \qquad k=1,2,3$$

† The results concerning real solutions are also included.

where the functions $\psi_k(\zeta)$ satisfy the system of ODE

$$\psi_k' = \mathrm{i}\left(\frac{c_k-\zeta}{c_l-\zeta}\right)^{\mathrm{i}\beta_l}\left(\frac{c_k-\zeta}{c_j-\zeta}\right)^{\mathrm{i}\beta_j}\frac{\psi_l^*\psi_j^*}{(c_l-\zeta)(c_j-\zeta)} \qquad (14)$$

$$k\neq l\neq j \qquad k\neq j \qquad 1\leqslant l, j, k\leqslant 3.$$

Such a system, when $\beta_k=0$, was obtained in [3], where it was shown that for purely imaginary ψ_k the system (14) ($\beta_k=0$) reduces to PVI.

Herein we shall show that in the case $\beta_k=0$ the system (14) reduces to PVI in the general complex case.

First of all, define the new variables

$$\psi_1(\zeta)=\mathrm{i}\alpha_1(c_3-c_2)^{\mathrm{i}\beta_1}\tilde{w}_1(\hat{z}) \qquad \psi_2(\zeta)=\mathrm{i}\alpha_2(c_3-c_1)^{\mathrm{i}\beta_2}\hat{z}^{\mathrm{i}\beta_2}\tilde{w}_2(\hat{z})$$

$$\psi_3(\zeta)=\mathrm{i}\alpha_3(c_2-c_1)^{\mathrm{i}\beta_3}\left(\frac{\hat{z}}{1-\hat{z}}\right)^{\mathrm{i}\beta_3}\tilde{w}_3(\hat{z}) \qquad \hat{z}=\frac{c_3-c_2}{c_3-c_1}\,\frac{\zeta-c_1}{\zeta-c_2}$$

$$\alpha_1=\sqrt{(c_3-c_1)(c_2-c_1)} \qquad \alpha_2=-\sqrt{(c_3-c_2)(c_2-c_1)} \qquad \alpha_3=\sqrt{(c_3-c_2)(c_3-c_1)}.$$

In new variables, the system (14) takes the form

$$\begin{aligned}&(\hat{z}-1)\tilde{w}_1'=\tilde{w}_2^*\tilde{w}_3^* \qquad \hat{z}(\hat{z}-1)\tilde{w}_3'-\mathrm{i}\beta_3\tilde{w}_3=(1-\hat{z})\tilde{w}_1^*\tilde{w}_2^*\\ &\hat{z}(\hat{z}-1)\tilde{w}_2'+\mathrm{i}\beta_2(\hat{z}-1)\tilde{w}_2=\tilde{w}_1^*\tilde{w}_3^*.\end{aligned} \qquad (15)$$

To simplify the notation, we write z instead of $\hat{z}$, $\tilde{z}$, and w instead of $\hat{w}$, $\tilde{w}$. Moreover, put $\beta_k=0$. As a result we have the following system:

$$(z-1)w_1'=w_2^*w_3^* \qquad zw_3'=-w_1^*w_2^* \qquad z(z-1)w_2'=-\varepsilon w_1^*w_3^*. \qquad (16)$$

The case $\varepsilon=-1$ correspond to the system (15), and $\varepsilon=1$ corresponds to the system (13).

Thus, our goal is to show that general complex solutions of the system (16) may be presented in terms of PVI.

In analogy to what was done in the discussion of reduction I, we introduce the notation

$$w_k=\rho_k\exp(\mathrm{i}\varphi_k) \qquad k=1,2,3 \qquad \psi=\varphi_1+\varphi_2+\varphi_3.$$

There are two integrals of the system (16)

$$\rho_1^2+\varepsilon\rho_2^2+\rho_3^2=C_1 \qquad \rho_1\rho_2\rho_3\sin\psi=C_2. \qquad (17)$$

Using (17) the system (16) can be rewritten as follows:

$$\varphi_1'=\frac{C_2}{(1-z)p_1} \qquad \varphi_2'=\frac{-C_2}{z(1-z)p_2} \qquad \varphi_3'=\frac{C_2}{zp_3} \qquad p_k=\rho_k^2 \qquad (18)$$

$$p_1'=\frac{2C_2}{z-1}\cot\psi \qquad p_2'=-\frac{2C_2}{z(z-1)}\cot\psi \qquad p_3'=-\frac{2C_2}{z}\cot\psi \qquad (19)$$

$$\begin{aligned}&\left(\rho_1''+\frac{\rho_1'}{z-1}-\frac{\varepsilon\rho_1(\rho_1^2-C_1)}{2(z-1)^2}-\frac{C_2^2}{\rho_1^3(z-1)^2}\right)^2\left(\frac{z(z-1)}{z-2}\right)^2\\ &\qquad=\rho_1^2\left\{\left(\frac{\rho_1^2-C_1}{2(z-1)}\right)^2+\varepsilon\left[(\rho_1')^2+\frac{C_2^2}{\rho_1^2(z-1)^2}\right]\right\}.\end{aligned} \qquad (20)$$

For a given general solution ($\rho_1 > 0$) of equation (20), a general solution of the system (16) can be easily reconstructed. There is an indefiniteness in the case $C_2 = 0$ (see (17), (19)). It is to be uncovered by means of the second equation in (17); in other words, one has to make the following change in (19):

$$C_2 \cot \psi \xrightarrow[C_2 \to 0]{} \rho_1 \rho_2 \rho_3 .$$

In equations (18) and (20) in this case simply put $C_2 = 0$.

The case $C_2 = 0$ was considered in [3] ($\varepsilon = -1$) and in [1] ($\varepsilon = 1$). In this case one has to introduce new variables

$$\rho_1(z) = \sqrt{\varepsilon}\, \phi(\xi) \qquad z = 4\sqrt{\xi}(2\sqrt{\xi} - 1 - \xi)^{-1}. \tag{21}$$

Then for function $\phi(\xi)$ we obtain the particular case of equation (A1.2) (see appendix 1). Equation (A1.2) in turn is equivalent to PVI.

The case $C_2 \neq 0$ is not so trivial, as after substitution of (21) into (20) the resulting equation contains two more terms proportional to C_2^2 than appear in (A1.2). These terms cannot be removed by a known transformation.

To cope with this case we rewrite the system (16) in a slightly different form:

$$p_3 = C_1 - p_1 - \varepsilon p_2 \tag{22}$$

$$z(1-z)p_2' = 2\sqrt{p_1 p_2 p_3 - C_2^2} \qquad (z-1)p_1' = 2\sqrt{p_1 p_2 p_3 - C_2^2}. \tag{23}$$

The p_k are defined in the last equation of (18). Now, having substituted formula (22) into the subrooting expression (23), we introduce a new variable T:

$$\sqrt{-\varepsilon p_1 p_2^2 + p_1(C_1 - p_1)p_2 - C_2^2} = p_2 T + \mathrm{i} C_2 . \tag{24}$$

In (24) it is easy to recognise an Euler substitution well known in the theory of integration. It is necessary to mention that the situation we have differs from the standard one, as the quantity p_1 is not a constant, but a function of z.

Define another variable U:

$$U = -\varepsilon p_1 - T^2. \tag{25}$$

In new variables we have the following expression for p_2:

$$U p_2 = 2\mathrm{i} C_2 T + \varepsilon(U + T^2)[C_1 + \varepsilon(U + T^2)] \tag{26}$$

and the system (22), (23) may be rewritten as follows:

$$U' = \frac{-2\varepsilon}{z-1}\left(\mathrm{i}C_2 + T(\varepsilon C_1 + 2T^2) + \frac{2z-1}{z}\, TU\right) \tag{27}$$

$$T' = \frac{-\varepsilon}{U(z-1)}\left(2\mathrm{i}C_2 T + T^2(\varepsilon C_1 + T^2) + \frac{1-z}{z}\, U^2\right). \tag{28}$$

System (27), (28) appears to be equivalent to equation (A1.2). To see this, U has to be obtained from equation (28) and substituted into equation (27). Then we find that the function

$$\phi(\theta) = 2T(z) - \frac{\mu}{4} \qquad z = (1-\theta)^{-1}$$

satisfies equation (A1.2), and the coefficients λ, μ, ν, k^2 of this equation are given in terms of the parameters C_1, C_2 in the following way:

$$\lambda = \varepsilon \qquad \mu k^2 = 32\mathrm{i}C_2 \tag{29}$$

$$-\frac{k^2}{4} = \frac{k^2}{4} \mp \frac{k\mu}{4} + 2\varepsilon C_1 = \nu - \left(\frac{\mu}{2} \pm \frac{k}{2}\right)^2. \tag{30}$$

It is evident that multiplying the first equality in (30) by k and using the second equation in (29), we obtain a cubic equation which defines k if C_1 and C_2 are given. Further, with the help of (29) we shall find μ and, from the second equality in (30), find ν.

The choice of the sign (upper/lower) in (30) is arbitrary, but still the same in both formulae.

Appendix 1 contains the formulae (A1.5)-(A1.7), which present one-to-one correspondance between solutions of equation (A1.2) and PVI. Using the formulae (A1.3), (A1.4) of appendix 1, it is easy to express C_1 and C_2 directly in terms of the coefficients of PVI.

Note that asymptotic properties of PVI were studied in [10]. Besides that, information about partial solutions may be found in appendix 1, and also in [11].

In conclusion, it is necessary to mention that the transformation of equation (20) into equation (A1.2) found here allows one to obtain the new transformation for PVI. This transformation can be called the quadratic transformation in analogy with corresponding one for certain hypergeometric equations. Details will be published elsewhere.

Appendix 1. Comments on partial solutions

The canonical form of PVI is as follows [12]:

$$w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w'$$
$$+ \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right]. \tag{A1.1}$$

The following theorem and the method it provides for obtaining Bäcklund transformations were given in [13], see also [14]. To tell the truth, formulae (A1.5)-(A1.7) (see below) are equivalent to the system of equations obtained earlier in [15]. But in [15] they were not used to derive Bäcklund transformations.

Theorem [13]. Let us consider $w(z, \alpha, \beta, \gamma, \delta)$—a solution of equation (A1.1), and $\phi(z, \lambda, \mu, \nu, k)$—a solution of the following equation:

$$\begin{aligned}
&[z(z-1)\Omega]^2 = \Psi^2\Delta \\
&\Omega = \phi'' + \frac{3z-1}{2z(z-1)}\phi' + \frac{2\phi I + (\mu/2)I - k^2\phi}{z(z-1)^2} \\
&I = \phi^2 + \frac{\mu}{2}\phi + \nu \qquad \Delta = (\phi')^2 + \frac{I^2 - k^2\phi^2}{z(z-1)^2} \\
&\Psi = (z+1)\left(\phi + \frac{\mu}{4}\right) + \frac{\lambda}{2}(z-1)
\end{aligned} \tag{A1.2}$$

where the parameters λ, μ, ν, k are defined by the formulae

$$k=\sqrt{-2\beta}-\sqrt{2\alpha}-1 \qquad \lambda=\sqrt{-2\beta}+\sqrt{2\alpha} \tag{A1.3}$$

$$\frac{k\mu}{4}=\tfrac{1}{2}-\gamma-\delta \qquad \nu=2\delta-1+\left(\frac{\mu}{4}+\frac{k}{2}\right)^2. \tag{A1.4}$$

In formulae (A1.3) the branches of the roots are chosen arbitrary, but are the same in both formulae.

Assume $\Delta\neq 0$, $A\neq 0$, the formulae

$$w=\frac{1}{2A}\left(-\frac{z+1}{z}\phi'+\frac{2k}{z(z-1)}\phi-(z-1)^2\frac{\Omega}{\Psi}\right) \tag{A1.5}$$

$$A=-\frac{1}{z}\phi'+\frac{1}{2z^2}\left(I+\frac{k(z+1)}{z-1}\phi\right) \tag{A1.6}$$

$$\phi=z\frac{w'}{w}+\frac{\lambda-k-1}{2(z-1)}w+\frac{(\lambda+k+1)z}{2(z+1)w}-\frac{\lambda}{2}\frac{z+1}{z-1}-\frac{1}{2}-\frac{\mu}{4} \tag{A1.7}$$

provide a one-to-one correspondance between solutions of equations (A1.1) and (A1.2).

Remark. The works [13, 14] instead of the conditions $\Delta\neq 0$, $A\neq 0$ contain the conditions $k\neq 0$ and $\phi\neq 0$ when $\nu=0$. In [11] it is correctly mentioned that the restriction $k\neq 0$† is unnecessary, and instead of $\phi\neq 0$ when $\nu=0$, we should have the more general condition that $A\neq 0$. The condition $\Delta\neq 0$ is omitted everywhere. However, that is a necessary condition, since on the one hand if $\Delta\equiv 0$, then equation (A1.2), in agreement with the identity

$$2\phi'\Omega=\Delta'+\frac{3z-1}{z(z-1)}\Delta$$

is satisfied automatically, but on the other hand the condition $\Delta\equiv 0$ is incompatible with the formulae (A1.5)-(A1.7), except for the case when $\phi=$ constant and satisfies

$$I^2=k^2\phi^2. \tag{A1.8}$$

The condition (A1.8) needs separate verification, as it appears not only from the condition $\Delta\equiv 0$, but also in obtaining equation (A1.2), when it is necessary to reduce the multiplier $I^2-k^2\phi^2$. The result of this verification depends on k. If $k\neq 0$, then $w=1$ corresponds to a solution ϕ of the equation $I=k\phi$, and to a solution of the equation $I=-k\phi-w=z$. Equation (A1.1) is undefined if $w=1, z$.

Let us now take $k=0$. Define ϕ_0 as a solution of the quadratic equation

$$I=\phi_0^2+\frac{\mu}{2}\phi_0+\nu=0.$$

Having substituted ϕ_0 in the left-hand side of (A1.7), we obtain

$$zw'+\frac{\lambda-1}{2(z-1)}w^2-\left(\frac{\lambda}{2}\frac{z+1}{z-1}+\frac{1}{2}+\sqrt{1-2\delta}\right)w+\frac{(\lambda+1)z}{2(z-1)}=0 \tag{A1.9}$$
$$\lambda=2\sqrt{2\alpha}+1 \qquad \sqrt{-2\beta}-\sqrt{2\alpha}=1.$$

† In the case $k=0$, μ is arbitrary.

Equation (A1.9) defines a one-parameter family of solutions of equation (A1.1) with the following conditions on coefficients

$$\gamma+\delta=\tfrac{1}{2} \qquad \sqrt{-2\beta}-\sqrt{2\alpha}=1.$$

In the last equality we assumed that there is the choice of branches of the square roots $\sqrt{2\alpha}$ and $\sqrt{-2\beta}$ for which it is true.

A general solution of (A1.9) when $\lambda\neq 1$, expressible in terms of the general solution χ of a hypergeometric equation, is

$$w=z+\frac{z(z-1)}{\sqrt{2\alpha}}\frac{\chi'}{\chi} \tag{A1.10}$$

$$z(z-1)\chi''+(1+\sqrt{2\alpha}-\sqrt{1-2\delta})(z-1)\chi'-\sqrt{2\alpha}\sqrt{1-2\delta}\chi=0. \tag{A1.11}$$

Let us consider now the case $A=0$. It is the consequence of the proof of the theorem (see [13, 14]) that a separate discussion is required for the special case $\phi=0$ when $\nu=0$. Similarly as was done to obtain (A1.9), we prove that the equation†

$$zw'+\frac{\sqrt{2\alpha}}{z-1}w^2+\frac{\sqrt{-2\beta}}{z-1}z-\left(\frac{\lambda}{2}\frac{z+1}{z-1}+\frac{1}{2}+\frac{\mu}{4}\right)w=0 \tag{A1.12}$$

defines a one-parameter family of solutions of (A1.1). When $\sqrt{2\alpha}\neq 0$ this equation is expressible in terms of a general solution of the hypergeometric equation by the same formula (A1.10), in which χ satisfies not (A1.11), but the following equation:

$$z(z-1)\chi''+[(1+\sqrt{2\alpha}-\sqrt{1-2\delta})z-(\sqrt{-2\beta}-\sqrt{1-2\delta})]\chi'-\sqrt{2\alpha}\sqrt{1-2\delta}\chi=0.$$

Note that in the case $k=0$ the equations (A1.9) and (A1.12) coincide. One-parameter solutions of (A1.12) were mentioned earlier in [15].

If $A=0$, $\phi\neq 0$ the latter approach is of no use. If $\phi\neq 0$ satisfies the equation $A=0$, then it also satisfies equation (A1.2) in which $\lambda=k+1$. These solutions of (A1.2) are assumed to correspond to $w=\infty$. Nevertheless these solutions may be used to generate one-parameter families of solutions of (A1.1). For this purpose it is necessary to change $k\to -k$ in formula (A1.6). This is possible because (A1.2) contains only k^2. As a result we have, if

$$\lambda=-k+1\Leftrightarrow\sqrt{-2\beta}=1 \qquad k\neq 0\Leftrightarrow\sqrt{2\alpha}\neq 0$$

then equation (A1.1) possesses a one-parameter family of solutions

$$w=\frac{1-z}{2\sqrt{2\alpha}}\left(\phi+\frac{\mu}{2}+\frac{\nu}{\phi}\right)+\frac{z+1}{2}$$

where ϕ a solution of the equation

$$\phi'=\frac{1}{2z}\left(I-\frac{\sqrt{2\alpha}(z+1)}{z-1}\phi\right). \tag{A1.14}$$

A general solution of (A1.14) is expressible in terms of a general solution χ of the hypergeometric equation

$$\phi=-\frac{\sqrt{2\alpha}}{2}-\frac{\mu}{4}+\sqrt{1-2\delta}-2z\frac{\chi'}{\chi}$$

$$\begin{aligned} &z(z-1)\chi''+[(1+\sqrt{2\alpha}-\sqrt{1-2\delta})z-(1-\sqrt{1-2\delta})]\chi' \\ &\qquad -\frac{\sqrt{2\alpha}}{2}\left(\frac{\sqrt{2\alpha}}{2}-\frac{\mu}{4}+\sqrt{1-2\delta}\right)\chi=0. \end{aligned} \tag{A1.15}$$

† To avoid μ, one uses formulae (A1.4) in which $\nu=0$.

This solution is discussed in [11]. But the connection with the hypergeometric equation (A1.13), (A1.15) was not found by the authors of [11]. Instead of the method above, they replaced ϕ in (A1.14) by (A1.17) and, having used (A1.1), obtained for w a nonlinear first-order ODE quadratic in w'. Furthermore, having applied the Lie-point symmetry of (A1.1) to the equation obtained by the method described above they obtain

$$(z(z-1)w')^2 = 2(w-z)^2[\alpha w^2 + (\beta+\gamma-\alpha)w - \beta] \qquad \delta = 0. \qquad \text{(A1.16)}$$

This equation was mentioned in [16]. Where it was used to define a one-parameter family of solutions to (A1.1) for $\delta = 0$. In [16], it was also shown that the transformation is, in fact, one of the Euler transformations, which maps (A1.16) to a linear ODE with four regular singular points—the Heun equation [17].

Thus our formulae allow one to show a non-trivial case of the integrability of the Heun equation in terms of hypergeometric functions. It turns out that there is a simpler trick for integrating equation (A1.16) in terms of hypergeometric functions. This trick is simply an application of the Euler transformation, but one other than that used in [16]. Here we write this transformation since, judging by [11], it has not previously been published

$$2(\alpha w^2 + (\beta+\gamma-\alpha)w - \beta) = (\sqrt{2\alpha}\, w + v)^2$$

whence

$$w = \frac{v^2 + 2\beta}{2\beta + 2\gamma - 2\alpha - 2\sqrt{2\alpha}\, v}.$$

Thus the defined function v satisfies the equation

$$2z(z-1)v' = \pm[v^2 + 2\beta - z(2\beta + 2\gamma - 2\alpha - 2\sqrt{2\alpha}\, v)]$$

which by the following change of variables:

$$v = \sqrt{-2\beta} + (-\sqrt{-2\beta} - \sqrt{2\alpha} + \sqrt{2\gamma})z \mp 2z(z-1)\frac{\chi'}{\chi}$$

maps into the hypergeometric equation

$$\begin{aligned} z(z-1)\chi'' &+ \{[2 \mp (\sqrt{2\gamma} - \sqrt{-2\beta} + \sqrt{2\alpha})]z - 1 \mp \sqrt{-2\beta}\}\chi' \\ &+ \tfrac{1}{4}(\sqrt{2\gamma} - \sqrt{-2\beta} - \sqrt{2\alpha})(\mp 2 + \sqrt{2\gamma} - \sqrt{-2\beta} + \sqrt{2\alpha})\chi = 0. \end{aligned}$$

It is clear that all partial solutions described above can be iterated by Bäcklund transformations and Lie-point symmetries. Some of these transformations together with the set of coefficients of (A1.1), which have one-parameter families of solutions, are expressible in terms of hypergeometric functions, as pointed out in [11].

Finally, it is necessary to mention that equation (A1.1) for $\alpha = \beta = \gamma = 0$, $\delta = \frac{1}{2}$ does not possess the solution, expressible in the superposition of hypergeometric and elliptic functions, at least in the way that was reported in [11]. In connection with this, apparently, theorem 7 of [11] is wrong.

Appendix 2. Bäcklund transformations

The method of obtaining Bäcklund transformations for (A1.1) by means of (A1.2) was given in [13]. The idea is as follows: after the transformation $k \to -k$, (A1.2) remains the same, but formulae (A1.3)-(A1.7) change. As a result, to one solution of (A1.2) there correspond two different solutions w and w_1 of (A1.1). Formulae (A1.5)-(A1.7) allow one to find a connection between them, i.e. a Bäcklund transformation, and formulae (A1.3), (A1.4)—the connection of coefficients of (A1.1) corresponding to these solutions.

As was mentioned in [11], (A1.2) remains the same after the transformation $\lambda \to -\lambda$, $\mu \to -\mu$, $\phi \to -\phi$. On the strength of this the new Bäcklund transformation of (A1.1) was obtained in [11].

The idea described above can be generalised as follows. For a given solution $w(z, \alpha, \beta, \gamma, \delta)$ of (A1.1) one obtains by means of (A1.7) and (A1.3), (A1.4) a solution of (A1.2)—$\phi(z, \lambda, \mu, \nu, k)$. From $\phi(z, \lambda, \mu, \nu, k)$ one passes to $\phi_1(z, \lambda_1, \mu_1, \nu_1, k_1)$ according to the formula

$$\phi_1(z, \lambda_1, \mu_1, \nu_1, k_1) = \varepsilon(\phi(z, \lambda, \mu, \nu, k) + a) \qquad \varepsilon^2 = 1. \tag{A1.16}$$

The parameters a, λ_1, μ_1, ν_1, k_1 are defined by explicit formulae in terms of λ, μ, ν, k (see below (A2.1)-(A2.6)).

Furthermore, one uses (A1.5), (A1.6)† to obtain $w_1(z, \alpha_1, \beta_1, \gamma_1, \delta_1)$, where $\alpha_1, \beta_1, \gamma_1, \delta_1$ are defined on a, λ_1, μ_1, ν_1, k_1 and at least on α, β, γ, δ by (A1.3), (A1.4). Thus, using a solution w of (A1.1), one obtains a new solution w_1 of this equation, corresponding, certainly, to a new set of coefficients.

Parameters λ, μ, ν, k and a, λ_1, μ_1, ν_1, k_1 are connected by the formulae listed below. Recall that they are defined by the condition that transformation (A1.16) maps solutions of (A1.2) into solutions of that equation, but possibly with new coefficients

$$a = 0 \qquad \varepsilon\lambda_1 = \lambda \qquad \varepsilon\mu_1 = \mu \qquad \nu_1 = \nu \qquad k_1 = -\varepsilon k \tag{A2.1}$$

$$a = 0 \qquad \varepsilon\lambda_1 = \lambda \qquad \mu_1 = \mu = 0 \qquad \nu_1 = -\nu \qquad k_1^2 = k^2 - 4\nu \tag{A2.2}$$

$$\forall a \qquad \varepsilon\lambda_1 = \lambda \qquad \varepsilon\mu_1 = \mu - 4a \qquad \nu_1 = \nu - \tfrac{1}{2}a\mu + a^2 \qquad k_1^2 = k^2 = 0 \tag{A2.3}$$

$$\begin{gathered} \forall a \qquad \varepsilon\lambda_1 = \lambda \qquad \varepsilon\mu_1 = -4a \qquad \mu = 0 \qquad \nu_1 + \nu = a^2 \\ k_1^2 = 0 \qquad k^2 = +4\nu \end{gathered} \tag{A2.4}$$

$$\begin{gathered} a \neq 0 \qquad \varepsilon\lambda_1 = \lambda \qquad \varepsilon\mu_1 = \mu - 4a \\ \nu_1 + \nu = a^2 \qquad k_1^2 = k^2 + a\mu - 4\nu \end{gathered} \tag{A2.5}$$

$$k_1^2 \neq 0 \qquad \mu a^2 - \left(\frac{\mu^2}{4} - k^2 + 4\nu\right) a + \mu\nu = 0. \tag{A2.6}$$

Formulae (A2.6) are related to (A2.5) only. In (A2.3) and (A2.4) a is arbitrary, but due to the condition $k_1 = 0$ it does not affect the Bäcklund transformation, i.e. for any a the transformation is the same as it is in the case $a = 0$. In other words, the transformation (A1.16) with parameters (A2.3) and (A2.4) defines the same Bäcklund transformations for (A1.1), as does transformations (A2.1) and (A2.2) in the case $k_1 = 0$.

Bäcklund transformations of (A1.1) corresponding to the transformation (A1.16) with the parameters (A2.17) were obtained in [13] for $\varepsilon = 1$ and in [11] for $\varepsilon = -1$. Thus, two new Bäcklund transformations are obtained in this paper. The first one is defined by formulae (A2.2), the second one by (A2.5) and (A2.6).

† It goes without saying that all parameters in these formulae are taken with subscript 1.

I shall not translate this transformation into the language in which equation (A1.1) is written. I advise the interested reader to do it himself as a simple exercise. It is quite clear that this sufficiently elementary procedure results in rather complicated formulae. It turns out that the 'spectral' interpretation of these transformations leads to more simple formulae. That is why a further study of these transformations is left for a more suitable case.

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